The regularity of ① – extremal – local minimizer
– minimizer
② – Weak solutions to E-L

is a huge topic!

We only focus on a very special case of ②

General cases please read Morrey's book or Giaquinta's book.

Since $\left(-L_{p_i}(\nabla u, u, x)\right)_{x_i} + L_z(\nabla u, u, x) = 0$

is a NONLINEAR PDE the result here is NOT covered by ch 6, where the regularity of a weak solution to a linear elliptic PDE was proved.

Ladyzhenskaya – Uralseva : Linear & quasi-linear elliptic PDEs

Gilbarg – Trudinger : 2nd order elliptic PDEs

Han – Lin : PDEs

contains a lot of discussions.

We do a <u>very special</u> case: $L = L(p)$

& the regularity in H^1, namely $W^{1,2}(\Omega)$.

A correction $u \in W^{1,q}(\Omega)$ is a weak solution iff

$$\int L_{p_i}(\nabla u, u, x)\, \varphi_{x_i} + L_z(\nabla u, u, x)\, \varphi = 0$$

$\forall \varphi \in C_c^\infty(\Omega)$. [unlike what I said. Confused by Condition (36) in Ch 8.2.3. They can be weaken!]

Namely using $\varphi \in C_c^\infty$ $\Rightarrow$ holds for $\varphi \in W_0^{1,q'}(\Omega)$

Theorem: Assume $u \in W^{1,2}(\Omega) = H^1(\Omega)$ is a weak solution to $\left(-L_{p_i}(Du)\right)_{x_i} = f$

$f \in L^2(\Omega)$. i.e. $\qquad L(p,u) = L(p) - fu$

$$\int_\Omega L_{p_i}(Du)\, \boxed{\varphi_{x_i}}^{\eta_{x_j}} = \int_\Omega f\cdot \varphi \qquad \forall \varphi \in H_0^1(\Omega).$$

$L(p)$

Assume further (1) $\|D_p^2 L\| \le C$,

(2) $L_{p_ip_j}\xi_i\xi_j \ge \theta |\xi|^2$

Then (1) $\forall V \subset\subset U\ (=\Omega)$ $\quad u \in H^2(V)$ $\sim C\left[\|f\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)}\right]$

effectively

$$\|D^2u\|_{L^2(V)} \le C\left[\|f\|_{L^2} + \|u\|_{H^1}\right].$$ — over $\Omega = U$

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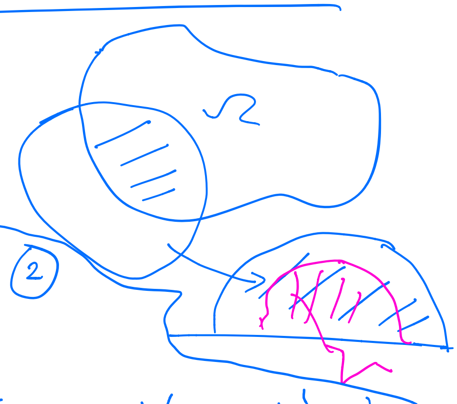
(2) If $u \in H_0^1(\Omega) \Longrightarrow u \in H^2(\Omega)$

$$\|u\|_{H^2} \le C\|f\|_{L^2}$$ — no u



(3) $\tilde{u} = D_{x_k}u$ satisfies $\leftarrow$ under (2)

$$(*) \quad \int L_{p_ip_j}\tilde{u}_{x_j}\varphi_{x_i} = \int f_{x_k}\varphi \qquad \forall \varphi \in H_0^1(\Omega)$$

Namely $\tilde{u}$ is a weak solution of the PDE

$$-\left(L_{p_ip_j}\tilde{u}_{x_j}\right)_{x_i} = \tilde{f} \qquad (\star)$$

Schauder

Proof: It uses the same idea of linear PDE regularity theory. i.e Use the difference quotient to estimate the derivative. — An idea of Nirenberg?

Main tool: Theorem 3 of ch 5.8

$$D_i^h u(x) := \frac{u(x+he_i)-u(x)}{h}$$ — Difference quotient.

$\forall x \in V, \quad 0<|h|< \mathrm{dist}(V, \partial\Omega)$.

(i) $1 \le p < +\infty \qquad u \in W^{1,p}(W)$

$\|D^h u\|_{L^p(V)} \le C \qquad (A)$

(ii) $1<p<\infty$ If (A) holds $\forall$ $0<|h|<\frac{1}{2}d(V, \partial\Omega)$

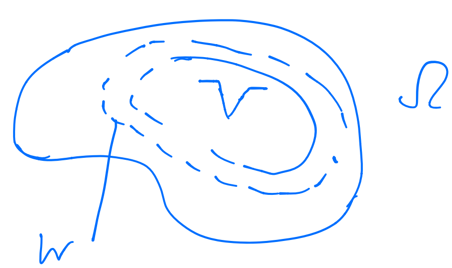
$\Rightarrow \quad u \in W^{1,p}(V)$ & $\|Du\|_{L^p} \le C$.

& $$\int u\, D_k^{-h} v = -\int D_k^h u \, v$$ Namely $\exists$ integration by parts for DQ

$$\int u(x) \frac{v(x-he_k)-v(x)}{-h} = \frac{1}{-h}\int u(x)\, \underbrace{v(x-he_k)}_{y} + \frac{1}{h}\int uv$$

$x = y + he_k$

$$= \frac{-1}{h}\int \left[u(x+he_k) - u(x)\right] v(x)$$



Let φ be a cut-off smooth function $\operatorname{supp}\varphi \supset W$ $\quad \varphi \equiv 1$ on W

$\in H_0^1(\Omega)$ $\quad \operatorname{supp}\varphi \subset \Omega$

$$v := \underline{-D_h^{-h}\left(\varphi^2 D_h^h u\right)} \qquad |h| \leq \operatorname{dist}(\operatorname{supp}\varphi, \partial\Omega)$$

Clearly, $v \in H_0^1(\Omega)$ — admissible testing function

in $\displaystyle\int L_{p_i}(Du)\, D_{x_i} v = \int f v$

$$\Longrightarrow \underbrace{\int \underbrace{D_k^h\left(L_{p_i}(Du)\right)}_{\text{LHS}} D_{x_i}\left(\varphi^2 D_k^h u\right)} = -\int \underline{f\, D_k^{-h}\left(\varphi^2 D_k^h u\right)}$$

Now we look into LHS.

The main difference is on how to handle

$$D_h^h\left(L_{p_i}(Du)\right)$$

$$= \frac{1}{h}\left(\underline{L_{p_i}\left(Du(x+he_k)\right) - L_{p_i}\left(Du(x)\right)}\right)$$

$$= \frac{1}{h}\int_0^1 \underline{\frac{d}{ds}\left(L_{p_i}\left(sDu(x+he_k) + (1-s)Du(x)\right)\right)}$$

$$= L_{p_ip_j}(\square) \qquad \square = sDu(x+he_k) + (1-s)Du(x)$$

$$= \frac{1}{h}\int_0^1 \underbrace{L_{p_ip_j}(\square) \cdot \left(\underbrace{D_j u(x+he_k) - D_j u(x)}\right)}$$

$$= \boxed{\underbrace{\int_0^1 L_{p_ip_j}(\square)\, ds}_{a_{ij}} \quad D_k^h D_j u}$$

$(a_{ij}) \geq \theta\, \mathrm{id}$

Hence

$$\mathrm{LHS} = \int_\Omega a_{ij}\, D_k^h D_j u\, D_{x_i}\left(\varphi^2 D_k^h u\right)$$

$$D_{x_i} u = D_i u$$

$$= \int_\Omega \underbrace{a_{ij}\, D_k^h D_j u\, D_k^h D_{x_i} u\, \varphi^2}_{\mathrm{I}} + \underbrace{a_{ij}\, D_k^h D_j u\, D_{x_i}(\varphi^2)\, D_k^h u}_{\mathrm{II}}$$

$$|D_{x_i}\varphi^2| = 2\varphi\, |D_{x_i}\varphi|$$

$$\mathrm{I} \geq \int_\Omega \theta\, |D_k^h Du|^2 \varphi^2$$

$$|\mathrm{II}| \leq \int_\Omega A_\varphi\, |D_k^h Du|\, |D_k^h u|\, |\varphi|$$

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$$

$$\leq \frac{\theta}{4} \int |D_k^h Du|^2 \varphi^2 + C(\theta, A) \int |D_k^h u|^2$$

$$\Rightarrow \mathrm{LHS} \geq \frac{3}{4}\theta \int |D_k^h Du|^2 \varphi^2 - C(\theta, A) \int |Du|^2 \cancel{|D\varphi|^2}$$

$$\|$$

$$\mathrm{RHS} = \int - f\, \underbrace{D_k^{-h}\left(\varphi^2 D_k^h u\right)}_{v}$$

$$\leq \int |f|\,|v| \leq \varepsilon \int |v|^2 + \frac{1}{4\varepsilon} \int |f|^2$$

$$\int |v|^2 \leq C \int \left| D(\varphi^2 D_k^h u) \right|^2$$

$$D_j D_k^h u = D_{x_j}\left(\frac{u(x+he_k) - u(x)}{h} \right)$$

$$= \frac{D_j u(x+he_k) - D_j u(x)}{h} = D_k^h D_j u$$

$$\leq C \int |(D\varphi^2)|^2 |D_k^h u|^2 + C \int \varphi^2 |D D_k^h u|^2$$

$$|D D_k^h u|^2 = |D_k^h D u|^2$$

$$\leq C_2 \int |Du|^2 + C_3 \int \varphi^2 |D_k^h D u|^2 \leftarrow \varepsilon$$

Hence if we choose $\varepsilon = \frac{\theta}{4} \cdot \frac{1}{C_3}$

$$RHS \leq \frac{\theta}{4} \int \varphi^2 |D_k^h D u|^2 + C(\theta, A)\left(\int |Du|^2 + \int |f|^2 \right)$$

Hence

$$\frac{\theta}{2} \int \varphi^2 |D_k^h D u|^2 \leq C(\theta, A)\left(\int |Du|^2 + \int |f|^2 \right)$$

We get the estimate

$$\int_W |D_k^h D u|^2 \leq C(\theta, A)\left(\int_\Omega |Du|^2 + \int_\Omega |f|^2 \right)$$

This proves (1) by Theorem 3 of ch 5.8

Note $p=2$ in this case.

Now we worry about (2)

1st the estimate can be done on boundary by boundary flatting as in ch 6.3

Namely

$$\boxed{\int_U |D^2u| \leq C\left(\int_U |f|^2 + \int_U |Du|^2\right)}$$

Now FT of calculus

$$(L_p(Du) - L_p(0))\cdot Du = \int_0^1 \frac{d}{ds}\left(L_p(sDu)\cdot Du\right)$$

$$= \int_0^1 L_{p_ip_j}^{(sDu)} D_{x_i}u\, D_{x_j}u \geq \theta |Du|^2$$

$$\Rightarrow L_p(Du)\cdot Du - \sum a_i D_{x_i}u \geq \theta |Du|^2$$

$$\Rightarrow \int f\cdot u = \int L_{p_i}^{(Du)} D_{x_i}u$$

$$\geq \theta \int |Du|^2 + \int_{\partial\Omega} \sum a_i D_{x_i}u$$

$$\int_{\partial u} u\, a_i\nu^i = 0$$

ν_i unit exterior normal

Hence $\Rightarrow$ $\theta\left(\int |Du|^2\right) \leq \int |f||u|$

$$\leq \left(\int |f|^2\right)^{\frac{1}{2}} \left(\int |u|^2\right)^{\frac{1}{2}}$$

$$\leq \Lambda\left(\int |f|^2\right)^{\frac{1}{2}} \left(\int |Du|^2\right)^{\frac{1}{2}}$$

$$\Rightarrow \boxed{\int |Du|^2 \leq C(\Lambda,\theta) \int |f|^2}$$

plus - Poincare inequality.

$$\boxed{\int |u|^2 \leq C \int |Du|^2} \quad b$$

$a + b \Rightarrow 2$

Higher regularity: Requires De Giorgi–Nash's result & Schauder's result $\leftarrow$ [Han-Lin used Morrey's proof]

Read 8.3.2 (No proofs)

Minimizing with constraints (ch 8.4)

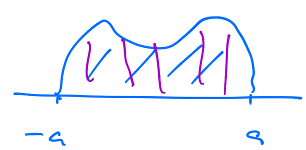
E.g.

a $\underbrace{I(u) = \frac{1}{2}\int_\Omega |Du|^2}_{\text{to be minimized or maximized}}$, $\quad J(u) = \int_\Omega G(u) = 0$

$\uparrow$ Constraint

e.g. Dido

$$\underbrace{I(u) = \int_{-a}^{a} u \, dx}$$



$u(-a) = u(a) = 0$, $\quad J(u) = \int_{-a}^{a} \sqrt{1 + \left(\frac{du}{dx}\right)^2}\, dx = \ell$

$\uparrow$ Fixed

- $\exists\ \lambda$ multiplier

(b) $I(u) = \frac{1}{2}\int_\Omega |Du|^2 - f \cdot u$

$A = \{ u \in H_0^1(\Omega) \mid u \geq h \text{ a.e in } \Omega \}$

minimizing with a barrier. – variational inequ

(c) Harmonic maps

$I(u) = \frac{1}{2}\int |Du|^2$ $\quad u: \Omega \to \mathbb{R}^L$

L-Large

$A = \{ u(x) \in N \subset \mathbb{R}^L \mid u \in H^1(\Omega, \mathbb{R}^L),\ u|_{\partial\Omega} = g \}$

a submanifold. — By Nash embedding this covers all cases

e.g. $N = S^{m-1} \subset \mathbb{R}^m$ is a typical case.

Typical in the sense $K_N > 0$

No typical since S^{m-1} has very special topology.

— Multiplier has geometric meaning not a $\lambda \in \mathbb{R}$ but a function

Stokes problem

(d) $\Omega \subset \mathbb{R}^3$ — Ω — Simply-connected → e.g. topologically trivial $\Omega \sim B$ equivalent to a ball.

$$\begin{cases} I(u) = \frac{1}{2}\int |Du|^2 - f \cdot u \\ A = \{ u \in H_0^1(\Omega, \mathbb{R}^3) \mid \operatorname{div}(u) = 0 \text{ in } \Omega \} \end{cases}$$

— multiplier is the pressure

There are many others: ↓ cohomology constraints — Applications of Hodge theory in $\mathbb{R}^3$

Riemannian - elliptic — Lorentz - hyperbolic

Hodge theory — Maxwell theory on $\mathbb{R}^{1,3}$

Gauge theory — Yang Mills theory on Lorentz

metric

1954

Two main concerns: ① existence of the constrained minimizer
② Lagrangian multiplier & its analytic meaning

ⓐ Theorem: $I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2$

$$A = \left\{ u \in H_0^1(\Omega),\ J(u) = \int_\Omega G(u) = 0 \right\}$$

e.g. $G(u) = \frac{1}{2}u^2 - c$, $G'(u) = g(u) = u$

Assume $g = G'$ satisfies $\begin{cases} |g(z)| \le C(1+|z|) \\ |G(z)| \le C(1+|z|^2) \end{cases}$

Then $\exists\, u \in A$ $I(u) = \inf\limits_{v\in A} I(v)$ (1)

Moreover it satisfies

(2) $\displaystyle\int \nabla u \cdot \nabla \varphi = \lambda \int g(u)\varphi$ for some $\lambda \in \mathbb{R}$ $\forall\, \varphi \in H_0^1(\Omega)$.

- Namely 1) the minimizer exists
2) It satisfies the Lagrangian multiplier type E-L.

Proof: Part 1). Pick u_k minimizing sequence

$I(u_k) \longrightarrow m = \inf\limits_{v\in A} I(v)$.

$\int |\nabla u_k|^2 \le A \implies \|u_k\|_{H^1} \le A$ by Poincaré ineq

$u_k \longrightarrow u$ in $H_0^1(\Omega)$ $\qquad \underline{\lim}_{k\to\infty} I(u_k) \geq I(u) = m$

We only need to show $u \in \mathcal{A}$ namely

$$\int G(u) = 0$$

But $$\int_\Omega \underbrace{G(u_k)}_{= 0} - G(u) = \int_\Omega \int_0^1 \frac{d}{dt}\big(G(\underbrace{tu_k + (1-t)u}_{u_t})\big) \qquad G'(u_t)(u_k - u)$$

$$= \int_\Omega \left\{\int_0^1 g(u_t)\,dt\right\}(u_k - u)$$

$$\Rightarrow \left| 0 - \int G(u) \right| \leq \int_\Omega C\,|u_k - u|\,\big[1 + |u_k| + |u|\big] \Rightarrow \longrightarrow 0$$

$\square$

For part (2) – assume $g(u) \neq 0$ a.e

Otherwise $G(u) = \text{const} = 0$ a.e $\Rightarrow$ $G(0) = 0$ since $u|_{\partial\Omega} = 0$

The minimizer $u = 0$ a.e Nothing to prove.

Now $\exists\, w \neq 0$, $w \in H_0^1(\Omega)$ $\displaystyle\int_\Omega \underbrace{g(u)}_{\in H_0^1(\Omega)}\, w \neq 0$

Now we construct a two-parameter variation

$$u + \tau v + \sigma w \qquad \forall v \in H_0^1(\Omega)$$

The condition $0 = J(\tau, \sigma) = \int_\Omega G(u + \tau v + \sigma w)$

$$J(0,0) = \int_\Omega G(u) := 0$$

$$\frac{\partial J}{\partial \tau}(0,0) = \int_\Omega G'(u)\, v = \int g(u)\, v$$

$$\frac{\partial J}{\partial \sigma}(0,0) = \boxed{\int_\Omega g(u)\, w \neq 0}$$

$\Rightarrow \exists\ \varphi$ such that $\varphi(0) = 0$

$$\boxed{0 = J(\tau, \sigma) = J(\tau, \varphi(\tau))} \qquad \left\{\begin{array}{l} \varphi : [-\varepsilon, \varepsilon] \to \mathbb{R} \\ \varphi(0) = 0 \quad (*) \\ C^1 \end{array}\right.$$

Namely $u + \tau v + \varphi(\tau) w \in \mathcal{A}$ for some φ

$$\Rightarrow \quad I(\tau) = \int \frac{1}{2} |\nabla u + \tau v + \varphi(\tau) w|^2$$

$$0 = I'(0) = \boxed{\int \langle \nabla u, \nabla v \rangle + \varphi'(0) \langle \nabla u, \nabla w \rangle}$$

But $(*) \Rightarrow \left.\frac{\partial J}{\partial \tau}\right|_{(0,0)} + \frac{\partial J}{\partial \sigma} \boxed{\varphi'(0)} = 0$

$0 = J(\tau, \varphi(\tau))$

$$\int_\Omega g(u)\, v + \varphi'(0) \int_\Omega g(u)\, w = 0$$

Then we have that

$$0 = \int \langle Du, Dv\rangle - \int \langle Du, Dw\rangle \int g(u)\, v \Big/ \int g(u)\, w$$

Let $\lambda = \int \langle Du, Dw\rangle \Big/ \int g(u)\, w$. $\square$

$v, u : \Omega \to \mathbb{R}^m$

$$\langle Du, Dv\rangle = \sum u^i_{x_j} (v)^i_{x_j} \qquad A = (u^i_{x_j})$$

$$= \operatorname{tran}\left({}^t(Dv) \cdot Du \right)$$

$$\frac{d}{dt} |Du|^2 = 2 \sum u^i_{x_j} u^i_{x_j t} \qquad \text{if } u(x, t) \text{ is a family}$$

If $|u|^2 = 1 \Rightarrow \sum_i u^i_{x_j} u^i = 0 \quad \forall j$

$${}^t u \cdot Du = 0 \quad \text{or} \quad {}^t(Du)(u) = 0$$

∇: $u \longrightarrow Du$ - vector valued $\qquad \Omega^0 \xrightarrow{d} \Omega^1$

Curl: $X \to \operatorname{curl}(X)$

$$(X^1, X^2, X^3) \longrightarrow \left(X^3_{x_2} - X^2_{x_3},\ X^3_{x_1} - X^1_{x_3},\ X^3_{x_1} - x^1_{x_2} \right) \qquad \Omega^1 \xrightarrow{d} \Omega^2$$

d- exterior $\qquad \Omega^2 \cong \Omega^1$

div: $X \longrightarrow \frac{\partial X^i}{\partial x_i}$ $\qquad \Omega^2 \xrightarrow{d} \Omega^3 \overset{*}{\cong} \Omega^0$

* also identify Ω^1 & Ω^2 $\qquad$ with a volume form